

NODAL SOLUTIONS FOR FOURTH ORDER ELLIPTIC EQUATIONS WITH CRITICAL EXPONENT ON COMPACT MANIFOLDS

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ABSTRACT. Using a variational method we prove the existence of nodal solutions to prescribed scalar Q -curvature type equations on compact Riemannian manifolds with boundary; these equations are fourth-order elliptic equations with critical Sobolev growth .

1. INTRODUCTION

The Paneitz operator was discovered by Paneitz [14] on 4-dimension manifolds and extended by Branson [4] to higher dimension ($n \geq 5$) is given by (1.1)

$$P_g^n(u) = \Delta_g^2 u - \operatorname{div}_g \left(\frac{(n-2)^2 + 4}{2(n-1)(n-2)} R_g g - \frac{4}{n-2} Ric_g \right)^\# du + \frac{n-4}{2} Q_g^n u$$

Where $\Delta_g u = -\operatorname{div}_g(\nabla u)$ is the Laplace-Beltrami operator, Ric_g is Ricci curvature, R_g is the scalar curvature of g , $\#$ is the musical isomorphism and Q_g^n is the Q -curvature which expresses as

$$Q_g^n = \frac{1}{2(n-1)} \Delta_g R + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R^2 - \frac{2}{(n-2)^2} |Ric_g|^2.$$

When (M, g) is Einsteinian manifold, the Paneitz-Branson operator has constant coefficients. It expresses as

$$P_g^n(u) = \Delta_g^2 u + \frac{n^2 - 2n - 4}{2n(n-1)} R_g \Delta_g u + \frac{(n-4)(n^2-4)}{16(n-1)^2} R_g^2 u.$$

In particular, when (M, g) is the unit n -sphere (S^n, h) , we can write

$$P_h^n(u) = \Delta_h^2 u + c_n \Delta_h u + d_n u$$

where $c_n = \frac{n^2-2n-4}{2}$ and $d_n = \frac{n(n-4)(n^2-4)}{16}$.

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The Paneitz-Branson operator is conformally invariant in the following sense: if $\tilde{g} = \varphi^{\frac{4}{n-4}}g$ is a metric conformal to g , for all $u \in C^\infty(M)$,

$$P_g^n(u\varphi) = \varphi^{\frac{n+4}{n-4}}P_g^n(u).$$

Taking $u = 1$, we find that

$$P_g^n\varphi = \frac{n-4}{2}Q_g^n\varphi^{\frac{n+4}{n-4}}.$$

We denote by $H_{2,0}^2(M)$ the standard Sobolev space which the intersection of $H_2^2(M)$ with $H_{1,0}^2(M)$. $H_{2,0}^2(M)$ will be endowed with the following suitable norm

$$(1.2) \quad \|u\|_{H_{2,0}^2(M)}^2 := \|\Delta u\|^2 + \|\nabla u\|^2 + \|u\|^2$$

Let (M, g) be a smooth Riemannian compact manifold with boundary of dimension n ($n \geq 5$). We let A be a smooth symmetric $(2, 0)$ -tensor on M and $a \in C^\infty(M)$. The Paneitz-Branson type operator with general coefficients is an operator of the form

$$(1.3) \quad P_g u = \Delta_g^2 u - \operatorname{div}_g [A^\# du] + au.$$

In this paper, we study the existence of a real positive number λ and a nodal solution u of the following Dirichlet problem

$$(1.4) \quad \begin{cases} P_g u = \lambda f |u|^{2^\sharp-2} u & \text{in } M \\ u = \phi_1, \partial_\nu u = \phi_2 & \text{on } \partial M \end{cases}$$

where P_g is Peinatz-Branson type operator (1.3), $\phi_1, \phi_2 \in C^\infty(M)$ are a boundary data and $2^\sharp = \frac{2n}{n-4}$ is the Sobolev critical exponent of the embedding $H_{2,0}^2(M) \hookrightarrow L^{2^\sharp}(M)$. When (ϕ_1, ϕ_2) change sign, u is called a nodal solution of the equation (1.4).

We assume in what follows that

- (1) P_g is coercive in the sense that there exists $\Lambda > 0$, such that for any $u \in H_{2,0}^2(M)$

$$\int_M (P_g u) u dv_g \geq \Lambda \|u\|_{H_{2,0}^2(M)}^2$$

- (2) f is the positive function in M
- (3) We have the Sobolev inequality (see [3]), for any $\epsilon > 0$ there is a constant B_ϵ such that for any $\varphi \in H_{2,0}^2(M)$ $\|\varphi\|_{2^\sharp}^2 \leq K_0(\epsilon + 1)\|\Delta_g \varphi\|_2^2 + B_\epsilon \|\varphi\|_2^2$, where K_0 is the best Sobolev's constant [8, 12, 13] given by

$$\frac{1}{K_0} = \frac{n(n^2 - 4)(n - 4)\omega_n^{\frac{4}{n}}}{16}$$

where ω_n denotes the volume of (\mathbb{S}, h) , the standard unit sphere of \mathbb{R}^{n+1} endowed with its round metric.

Now we cite some results concerning this problem. The problem of finding nodal solutions of second order equations has been studied by several authors:

In 1990, F.V. Atkinson, H. Brezis and L.A. Peletier [1], studied the existence of nodal solutions of the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u + |u|^p u & \text{in } B \\ u \neq 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

where B is the unit ball of \mathbb{R}^n ($n \geq 3$) $p = \frac{n+2}{n-2}$, and λ is a positive real number.

In 1994, E. Hebey and M. Vaugon [10] obtained results of existence and multiplicity of nodal solutions to the problem

$$\begin{cases} -\Delta_g u + au = f |u|^{\frac{4}{n-2}} u & \text{in } \Omega \\ u \neq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

where Ω is a smooth bounded domain of \mathbb{R}^n ($n \geq 3$), $a, f \in C^\infty(\bar{\Omega})$ and g is a Riemannian metric defined in a neighbourhood of $\bar{\Omega}$. Moreover they studied this problem when the zero Dirichlet condition on the boundary is replaced by a non-zero Dirichlet one.

In [11] D. Holcman obtained by variational method nodal solutions to the problem

$$\begin{cases} \Delta_g u + au = \lambda f |u|^{\frac{4}{n-2}} u & \text{in } M \\ u = \phi & \text{on } \partial M \end{cases}$$

where (M, g) is a smooth compact Riemannian manifold with boundary and of dimension $n \geq 3$, $a, f \in C^\infty(M)$, $\phi \in C^\infty(\partial M)$ is a changing sign and where λ is real positive number.

In 2002, Z. Djadli and A. Jourdain [7] studied the nodal solutions for scalar curvature type equations with a perturbation.

In this paper, we extend the results obtained by D. Holcman in [11] where the second order operator is replaced by a fourth order Paneitz-Branson type operator. The main results of this paper are stated as follows:

Theorem 1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$, with boundary $\partial M \neq \emptyset$. Let A be a smooth symmetric $(2, 0)$ -tensor and $a, f \in C^\infty(M)$, $f > 0$ and $x_0 \in M$ such that $f(x_0) = \max_M f$. We assume that the operator $P_g := \Delta_g^2 - \operatorname{div}_g (A(\nabla \cdot)^\#) + a$ is coercive and*

$$8(n-1) \operatorname{Tr}_g A(x_0) + (n-6)(n-4)(n+2) \frac{\Delta f(x_0)}{f(x_0)} - 4(n^2 - 2n - 4) R_g(x_0) < 0.$$

Then there exists a positive real number λ and a nontrivial solution $u \in H_{2,0}^2(M) \cap C^4(M)$ of the equation (1.4) which is a minimizer of the functional I defined on $H_{2,0}^2(M)$ by

$I(u) = \int_M \left((\Delta_g u)^2 + A \left((\nabla u)^\# , (\nabla u)^\# \right) + au^2 \right) dv_g$ under the constraint $\int_M f |u + h|^{2^\sharp} dv_g = \gamma$, where h denotes the unique solution of the problem

$$\begin{cases} \Delta_g^2 h - \operatorname{div}_g \left(A (\nabla h)^\# \right) + ah = 0 & \text{in } M \\ h = \phi_1, \partial_\nu h = \phi_2 & \text{on } \partial M \end{cases}.$$

The proof of theorem 1 proceeds in several steps: in a first section we use the approach developed by T. Aubin in [2]: we construct a minimizing sequence of solutions to the subcritical equations, in the second one, we show that this sequence converges weakly to a solution of the critical equation when the subcritical exponent tends to critical Sobolev exponent. In the third section we show that under some conditions we obtain a non trivial solution of the critical equation. The last section is devoted to test functions which verify the conditions assumed in the generic Theorem.

2. SUBCRITICAL SOLUTIONS

First, we have the following useful proposition.

Proposition 1. *Let (M, g) be a smooth Riemannian compact manifold with a smooth boundary of dimension $(n \geq 6)$ we assume that the operator P_g defined in (1.3) is coercive, then there exists a unique $h \in C^4(M)$ solution of*

$$(2.1) \quad \begin{cases} P_g h = 0 & \text{in } M \\ h = \phi_1, \partial_\nu h = \phi_2 & \text{on } \partial M \end{cases}.$$

Proof. Since the operator P_g is coercive, the existence and the uniqueness of the solution are guaranteed by the Lax-Milgram's theorem. The regularity follows from general regularity theory. \square

Letting $w = u - h$, we observe that $u \in C^4(M)$ is solution of the equation (1.4) if and only if $w \in C^4(M)$ is solution of

$$(2.2) \quad \begin{cases} P_g w = \lambda f |w + h|^{2^\sharp - 2} (w + h) & \text{in } M \\ w = \partial_\nu w = 0 & \text{on } \partial M \end{cases}.$$

The subcritical problem associated to 2.2 is then

$$(2.3) \quad \begin{cases} P_g w = \lambda f |w + h|^{q-2} (w + h) & \text{in } M \\ w = \partial_\nu w = 0 & \text{on } \partial M \end{cases}$$

where $2 < q < 2^\sharp = \frac{2n}{n-4}$. The functional on $H_{2,0}^2(M)$ associated to equation (2.3) is defined by

$$I(w) = \int_M w P_g w dv_g = \int_M \left((\Delta w)^2 + A \left((\nabla w)^\# , (\nabla w)^\# \right) + aw^2 \right) dv_g, \quad q \in \left(2, 2^\sharp \right).$$

Let

$$\mu_{\gamma,q} := \inf_{w \in \mathcal{H}_q} I(w), \quad \text{and } \mathcal{H}_q = \left\{ w \in H_{2,0}^2(M) \text{ such that } \int_M f |w + h|^q dv_g = \gamma \right\}$$

where γ denotes some constant such that

$$(2.4) \quad \int_M f |h|^{2^\sharp} dv_g < \gamma.$$

First we claim that

Lemma 1. *For all $\gamma > 0$ and for all $0 \leq q < 2^\sharp$, there exists a real number $\lambda_{\gamma,q}$ and a function $w_{\gamma,q} \in \mathcal{H}_q$ solution to the problem (2.3).*

Proof. First we prove that \mathcal{H}_q is not empty, when the condition $\int_M f |h|^{2^\sharp} dv_g < \gamma$ is satisfied. To do so we put

$$F(t) = \int_M f |t\psi_1 + h|^q dv_g$$

where ψ_1 is the eigenfunction corresponding to the first eigenvalue λ_1 of Δ_g^2 such that

$$\begin{cases} \Delta_g^2 \psi_1 = \lambda_1 \psi_1 & \text{in } M \\ \psi_1 = \partial_\nu \psi_1 = 0 & \text{on } \partial M \end{cases}.$$

Obviously, we have

$$F(0) = \int_M f |h|^q dv_g < \gamma \text{ and } \lim_{t \rightarrow +\infty} F(t) = +\infty.$$

By continuity of F there exists $t_q > 0$ such that

$$F(t_q) = \int_M f |t_q \psi_1 + h|^q dv_g = \gamma.$$

Hence $t_q \psi_1 \in \mathcal{H}_q$.

Secondly we verify that $\mu_{\gamma,q}$ is finite. Let $u \in H_{2,0}^2(M)$. Since the tensor A is smooth, there exists $C > 0$ such that

$$(2.5) \quad \left| \int_M A((\nabla u)^\#, (\nabla u)^\#) dv_g \right| \leq C \int_M |\nabla u|^2 dv_g.$$

Moreover from the lemma 2.2 (see [15] page 16) for any $\eta > 0$ there exists $C(\eta) > 0$ such that for any $u \in H_{2,0}^2(M)$

$$(2.6) \quad \|\nabla u\|_2^2 \leq \eta \|\Delta u\|_2^2 + C(\eta) \|u\|_2^2.$$

Plugging (2.6) in (2.5) we get

$$\left| \int_M A((\nabla u)^\#, (\nabla u)^\#) dv_g \right| \leq C(\eta) \|\Delta u\|_2^2 + C'(\eta) \|u\|_2^2.$$

Using this last inequality and the expression of I , we derive that

$$(2.8) \quad I(u) \geq (1 - C(\eta)) \|\Delta u\|_2^2 - (C'(\eta) + \|a\|_\infty) \|u\|_2^2$$

where $\|\cdot\|_\infty$ is the supremum norm.

On the other hand, for all $u \in \mathcal{H}_q$, by Hölder's inequality we get that

$$(2.9) \quad \|u\|_2^2 \leq Vol_g(M)^{1-\frac{2}{q}} \left(\left(\min_M f \right)^{-\frac{1}{q}} \gamma^{\frac{1}{q}} + \|h\|_q \right)^2.$$

Plugging (2.9) in (2.8) and taking η small enough, we get

$$1 - C(\eta) = C_1 > 0.$$

So

$$(2.10) \quad I(u) \geq C_1 \|\Delta u\|_2^2 + C_2$$

where C_1 some positive constant and C_2 is a constant independent of u .

Let (w_i) be a minimizing sequence of the functional I on \mathcal{H}_q . Hence for i sufficiently large $I(w_i) \leq \mu_{\gamma,q} + 1$ and by (2.10) we derive that

$$\|\Delta w_i\|_2^2 \leq \frac{1}{C_1} (\mu_{\gamma,q} + 1 - C_2).$$

By formula (2.6) and (2.9) we obtain that $\|\nabla w_i\|_2^2$ and $\|w_i\|_2^2$ are bounded. So the sequence (w_i) is bounded in $H_{2,0}^2(M)$ and by the reflexivity of this latter space there exists subsequence of (w_i) still denoted (w_i) such that

$$\begin{aligned} \cdot \quad & w_i \rightharpoonup w_{\gamma,q} \text{ weakly in } H_{2,0}^2(M) \\ \cdot \quad & w_i \rightarrow w_{\gamma,q} \text{ strongly in } H_1^2(M) \text{ and in } L^s \text{ for any } s < 2^\# \\ \cdot \quad & \|w_{\gamma,q}\|_{H_{2,0}^2} \leq \liminf_i \|w_i\|_{H_{2,0}^2}. \end{aligned}$$

Consequently

$$\begin{aligned} I(w_{\gamma,q}) &= \int_M \left((\Delta w_{\gamma,q})^2 + A \left((\nabla w_{\gamma,q})^\# , (\nabla w_{\gamma,q})^\# \right) + a w_{\gamma,q}^2 \right) dv_g \\ &\leq \liminf_i \|\Delta w_i\|_2^2 + \lim_i \int_M A \left((\nabla w_i)^\# , (\nabla w_i)^\# \right) dv_g + \lim_i \int_M a w_i^2 dv_g \\ &= \lim_i I(w_i) = \mu_{\gamma,q} \end{aligned}$$

and since

$$\int_M f |w_{\gamma,q} + h|^q dv_g = \lim_i \int_M f |w_i + h|^q dv_g = \gamma$$

we obtain

$$I(w_{\gamma,q}) = \mu_{\gamma,q}$$

so $w_{\gamma,q}$ satisfies

$$\begin{aligned} & \int_M \left(\Delta w_{\gamma,q} \Delta \varphi + A \left((\nabla w_{\gamma,q})^\# , (\nabla \varphi)^\# \right) + a w_{\gamma,q} \varphi \right) dv_g \\ &= \lambda_{\gamma,q} \int_M f |w_{\gamma,q} + h|^{q-2} (w_{\gamma,q} + h) \varphi dv_g \end{aligned}$$

for any $\varphi \in H_{2,0}^2(M)$; where $\lambda_{\gamma,q}$ is the Lagrange multiplier. Hence $w_{\gamma,q}$ is a weak solution of the equation

$$\begin{cases} \Delta_g^2 w_{\gamma,q} - \operatorname{div} A(\nabla w_{\gamma,q})^\# + a w_{\gamma,q} = \lambda_{\gamma,q} f |w_{\gamma,q} + h|^{q-2} (w_{\gamma,q} + h) & \text{in } M \\ w_{\gamma,q} = \partial_\nu w_{\gamma,q} = 0. & \text{on } \partial M \end{cases}$$

Using the bootstrap method, we show that $w_{\gamma,q} \in L^s(M)$ for any $(s < 2^\#)$, so $P_g(w_{\gamma,q}) = \Delta_g^2 w_{\gamma,q} - \operatorname{div} A(\nabla w_{\gamma,q})^\# + a w_{\gamma,q} \in L^s(M)$ for any $(s < 2^\#)$ and since P_g is a fourth order elliptic operator, it follows by a well known

regularity theorem that $P_g(w_{\gamma,q}) \in C^{0,\alpha}(M)$, for some $\alpha \in (0,1)$ then $w_{\gamma,q} \in C^{4,\alpha}(M)$. \square

3. CRITICAL SOLUTIONS

In this section one will study the behavior of the sequence $w_{\gamma,q}$ when q goes to critical Sobolev exponent 2^\sharp .

Lemma 2. *Under the hypothesis $\int_M f |h|^{2^\sharp} dv_g < \gamma$, the sequence $(w_{\gamma,q})_q$ is bounded in $H_{2,0}^2(M)$. The Lagrangian multipliers $\lambda_{\gamma,q}$ are strictly positive and the sequence $(\lambda_{\gamma,q})_q$ is bounded when q goes to 2^\sharp .*

Proof. First we have

$$\begin{aligned} 0 &\leq \mu_{\gamma,q} = \int_M (P_g w_{\gamma,q}) w_{\gamma,q} dv_g \\ &= \lambda_{\gamma,q} \int_M f |w_{\gamma,q} + h|^{q-2} (w_{\gamma,q} + h) w_{\gamma,q} dv_g \\ &= \lambda_{\gamma,q} \left(\gamma - \int_M f |w_{\gamma,q} + h|^{q-2} (w_{\gamma,q} + h) h dv_g \right). \end{aligned}$$

By Hölder's inequality we get

$$\int_M f |w_{\gamma,q} + h|^{q-2} (w_{\gamma,q} + h) h dv_g \leq \gamma^{1-\frac{1}{q}} \left(\int_M f |h|^q \right)^{\frac{1}{q}} < \gamma.$$

So, we deduce that $\lambda_{\gamma,q} \geq 0$. Moreover if $\lambda_{\gamma,q} = 0$, then $w_{\gamma,q} = 0$. Hence a contradiction with the fact that $w_{\gamma,q} \in \mathcal{H}_q$ and $\int_M f |h|^{2^\sharp} dv_g < \gamma$.

Now, we prove that the sequence $(w_{\gamma,q})_q$ is bounded in $H_{2,0}^2(M)$.

Let ψ_1 be an eigenfunction of the Δ_g^2 related to the eigenvalue λ_1 such that

$$\begin{cases} \Delta_g^2 \psi_1 = \lambda_1 \psi_1 & \text{in } M \\ \psi_1 = \partial_\nu \psi_1 = 0 & \text{on } \partial M \\ \int_M \psi_1^2 dv_g = 1 \end{cases}.$$

Letting

$$F(t_q) = \int_M f |t\psi_1 + h|^q dv_g$$

we have that $F(t_q) = \gamma$.

We will show that $\frac{\partial F(t_q, q)}{\partial t} \neq 0$: we argue by contradiction. Suppose that $\frac{\partial F(t_q, q)}{\partial t} = 0$. Obviously, we have

$$(3.1) \quad t_q \frac{\partial F(t_q, q)}{\partial t} = \gamma - \int_M f |t\psi_1 + h|^{q-2} (t\psi_1 + h) h dv_g = 0$$

and using the Hölder's inequality, we deduce

$$\int_M f |t\psi_1 + h|^{q-2} (t\psi_1 + h) h dv_g \leq \left(\int_M f |t\psi_1 + h|^q dv_g \right)^{1-\frac{1}{q}} \left(\int_M f |h|^q dv_g \right)^{\frac{1}{q}} < \gamma.$$

Which contradicts (3.1).

Since $\frac{\partial F(t_q, q)}{\partial t} \neq 0$, by the implicit function theorem we get that t_q is continuous as a function q . Hence there exists a constant $C(\gamma)$ independent of q such that

$$(3.2) \quad \int_M w_{\gamma, q} P_g w_{\gamma, q} dv_g \leq I(t_q \psi_1) = t_q I(\psi_1) \leq C(\gamma).$$

With the coercivity of P_g , we get that the sequence $(w_{\gamma, q})_q$ is bounded in $H_{2,0}^2(M)$ when $q \rightarrow 2^\sharp$.

So we can extract a subsequence of $(w_{\gamma, q})$ still denoted $w_{\gamma, q}$, such that

- (a) $w_{\gamma, q} \rightarrow w$ weakly in $H_{2,0}^2(M)$ as $q \rightarrow 2^\sharp$
- (b) $w_{\gamma, q} \rightarrow w$ strongly in $H_1^2(M)$ and $L^s(M)$ for any $s < 2^\sharp$ as $q \rightarrow 2^\sharp$
- (c) $w_{\gamma, q} \rightarrow w$ a.e in M as $q \rightarrow 2^\sharp$.

Now, we prove that the Lagrange multiplier $\lambda_{\gamma, q}$ is bounded when $q \rightarrow 2^\sharp$.

By the formula (3.2) and the fact that

$$\int_M f |h|^q dv_g < \gamma$$

we obtain

$$\begin{aligned} 0 < \lambda_{\gamma, q} &= \frac{\int_M (P_g w_{\gamma, q}) w_{\gamma, q} dv_g}{\gamma - \int_M f |w_{\gamma, q} + h|^{q-2} (w_{\gamma, q} + h) h dv_g} \\ &\leq \frac{I(t_q \psi_1)}{\gamma - \gamma^{1-\frac{1}{q}} \int_M f |h|^q dv_g} < C(\gamma, h). \end{aligned}$$

Now, since $(\lambda_{\gamma, q})_q$ is bounded, then there is a subsequence of $(\lambda_{\gamma, q})_q$ still labelled $(\lambda_{\gamma, q})_q$ which converges to λ .

Putting

$$v_{\gamma, q} = w_{\gamma, q} - w \in H_{2,0}^2(M)$$

and taking into account of

$$\begin{aligned} \lim_{q \rightarrow 2^\sharp} \int_M \Delta w \Delta v_{\gamma, q} dv_g &\rightarrow 0 \\ \lim_{q \rightarrow 2^\sharp} \int_M A \left((\nabla w)^\sharp, (\nabla v_{\gamma, q})^\sharp \right) dv_g &\rightarrow 0 \\ \lim_{q \rightarrow 2^\sharp} \int_M a w v_{\gamma, q} dv_g &\rightarrow 0 \end{aligned}$$

we infer that

$$I(w_{\gamma, q}) = I(w) + I(v_{\gamma, q}) + 2 \int_M \left(\Delta w \Delta v_{\gamma, q} + A \left((\nabla w)^\sharp, (\nabla v_{\gamma, q})^\sharp \right) + a w v_{\gamma, q} \right) dv_g$$

$$(3.3) \quad I(w_{\gamma, q}) = I(w) + \|v_{\gamma, q}\|_2^2 + o(1)$$

By definition of μ i.e. $\mu := \lim_q I(w_{\gamma, q})$ we have

$$(3.4) \quad I(w) \geq \mu \text{ and } I(w_{\gamma, q}) = \mu + o(1).$$

And according to (3.3) and (3.4), we obtain

$$\|v_{\gamma, q}\|_2^2 = o(1).$$

Consequently $(v_{\gamma,q})_q$ converges strongly to 0 in $H_{2,0}^2(M)$ and $(w_{\gamma,q})_q$ converges strongly to w in $H_{2,0}^2(M)$. We conclude that $w \in \mathcal{H}_{2^\sharp}$ is a non trivial solution of the equation

$$\begin{cases} P_g w = \lambda f |w + h|^{2^\sharp-2} (w + h) & \text{in } M \\ w = \partial_\nu w = 0 & \text{on } \partial M \end{cases}.$$

□

Now we put $u := w + h$. If $(\phi_1, \phi_2) \neq (0, 0)$, then $u \neq 0$ is a non trivial solution of the equation

$$\begin{cases} P_g u = \lambda f |u|^{2^\sharp-2} u & \text{in } M \\ u = \phi_1 \text{ and } \partial_\nu u = \phi_2 & \text{on } \partial M \end{cases}.$$

And if $(\phi_1, \phi_2) \equiv (0, 0)$, then $h \equiv 0$ i.e. $u = w$. We will prove that under some condition, u is non trivial solution of the equation

$$(3.5) \quad \begin{cases} P_g u = \lambda f |u|^{2^\sharp-2} u & \text{in } M \\ u = \partial_\nu u = 0 & \text{on } \partial M \end{cases}$$

Proposition 2. *Suppose that the minimizing sequence $(w_{\gamma,q})_q$ converges weakly to w and put $\mu = \lim_q \mu_{\gamma,q}$. Assume that*

$$(3.6) \quad \mu < \frac{\gamma^{\frac{2}{2^\sharp}}}{K_0 \|f\|_\infty^{\frac{2}{2^\sharp}}}$$

then w is non trivial solution of the equation (3.5).

Proof. First, we have

$$\gamma^{\frac{2}{q}} = \left(\int_M f |w_{\gamma,q}|^q \right)^{\frac{2}{q}} \leq \|f\|_\infty^{\frac{2}{q}} Vol_g(M)^{\frac{2}{q} - \frac{2}{2^\sharp}} \left(\int_M |w_{\gamma,q}|^{2^\sharp} \right)^{\frac{2}{2^\sharp}}$$

and using the Sobolev inequality, we get for any $\epsilon > 0$, the existence of $B_\epsilon > 0$ such that

$$\begin{aligned} \gamma^{\frac{2}{q}} \|f\|_\infty^{-\frac{2}{q}} Vol_g(M)^{\frac{2}{2^\sharp} - \frac{2}{q}} &\leq (K_0 + \epsilon) \|\Delta w_{\gamma,q}\|_2^2 + B_\epsilon \|w_{\gamma,q}\|_2^2 \\ &\leq (K_0 + \epsilon) \left[(1 + \bar{\eta}) \|\Delta w_{\gamma,q}\|_2^2 - \bar{\eta} \|\Delta w_{\gamma,q}\|_2^2 \right] + B_\epsilon \|w_{\gamma,q}\|_2^2 \end{aligned}$$

where $\bar{\eta}$ is some small enough constant.

Now since

$$\|\Delta w_{\gamma,q}\|_2^2 = \mu_{\gamma,q} - \int_M \left(A \left((\nabla w_{\gamma,q})^\# , (\nabla w_{\gamma,q})^\# \right) + a w_{\gamma,q}^2 \right) dv_g$$

it follows that

$$\begin{aligned} &\gamma^{\frac{2}{q}} \|f\|_\infty^{-\frac{2}{q}} Vol_g(M)^{\frac{2}{2^\sharp} - \frac{2}{q}} \\ &\leq (K_0 + \epsilon) \left\{ (1 + \bar{\eta}) \left[\mu_{\gamma,q} - \int_M \left(A \left((\nabla w_{\gamma,q})^\# , (\nabla w_{\gamma,q})^\# \right) + a w_{\gamma,q}^2 \right) dv_g \right] \right. \\ &\quad \left. - \bar{\eta} \|\Delta w_{\gamma,q}\|_2^2 \right\} \\ &+ B_\epsilon \|w_{\gamma,q}\|_2^2. \end{aligned}$$

Since A is smooth, then for any $\eta > 0$ there exists $C(\eta) > 0$ such that

$$\left| \int_M A \left((\nabla w_{\gamma,q})^\# , (\nabla w_{\gamma,q})^\# \right) dv_g \right| \leq \eta \|\Delta w_{\gamma,q}\|_2^2 + C(\eta) \|w_{\gamma,q}\|_2^2$$

so we obtain that

$$\begin{aligned} & \gamma^{\frac{2}{q}} \|f\|_\infty^{-\frac{2}{q}} Vol_g(M)^{\frac{2}{2^\sharp} - \frac{2}{q}} - (K_0 + \epsilon)(1 + \bar{\eta}) \mu_{\gamma,q} \\ & \leq (K_0 + \epsilon) \left\{ (1 + \bar{\eta}) \left[\eta \|\Delta w_{\gamma,q}\|_2^2 + C(\eta) \|w_{\gamma,q}\|_2^2 + \|a\|_\infty \|w_{\gamma,q}\|_2^2 \right] \right. \\ & \quad \left. - \bar{\eta} \|\Delta w_{\gamma,q}\|_2^2 \right\} \\ & + B_\epsilon \|w_{\gamma,q}\|_2^2 \end{aligned}$$

and taking $\eta > 0$ such that

$$\bar{\eta} = \frac{\eta}{1 - \eta}$$

we get

$$\gamma^{\frac{2}{q}} \|f\|_\infty^{-\frac{2}{q}} Vol_g(M)^{\frac{2}{2^\sharp} - \frac{2}{q}} - (K_0 + \epsilon)(1 + \bar{\eta}) \mu_{\gamma,q} \leq C(\epsilon, \eta) \|w_{\gamma,q}\|_2^2$$

When $q \rightarrow 2^\sharp$ and the constants ϵ, η are chosen sufficiently small and if

$$\mu < \frac{\gamma^{\frac{2}{2^\sharp}}}{K_0 \|f\|_\infty^{\frac{2}{2^\sharp}}}$$

we infer that

$$\|w\|_2^2 \geq C' > 0.$$

Hence $w \not\equiv 0$. □

Now we are going to establish the regularity of the solution of the equation (2.2). We adapt the technique developed by Van der Vorst [16], Djadli-Hebey-Ledoux [6] and Esposito-Robert [15], for fourth-order elliptic equation.

First, we recall some useful results see [15].

Theorem 2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$, let $a \in C^\infty(M)$ and let A be smooth symmetric $(2, 0)$ tensor on M . Assume that the operator $P_g := \Delta_g^2 - \text{div}_g A(\nabla \cdot)^\# + a$ is a coercive, then for any $f \in H_k^p(M)$ there exists a unique $u \in H_{4+k}^p(M)$ such that $P_g u = f$. Moreover we have*

$$\|u\|_{H_{4+k}^p(M)} \leq C \|f\|_{H_k^p(M)}$$

Theorem 3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 1$, let $p \geq 1$ and let $0 \leq m \leq k$ two integers such that $n \geq p(k - m)$, then $H_k^p(M)$ is embedded in $H_m^q(M)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k-m}{n}$*

Our regularity theorem states as follows

Theorem 4. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Assume that the operator $P_g := \Delta_g^2 - \operatorname{div}_g A(\nabla \cdot)^\# + a$ is coercive. Let $u \in H_2^2(M)$ be a weak solution of equation then $u \in C^4(M)$ and u is a strong solution to equation (2.2).*

Proof. Let $u \in H_2^2(M)$ be a weak solution of (2.2). We proceed as in Van der Vorst [16], and we claim that for any $\epsilon > 0$, there exists $q_\epsilon \in L^{\frac{n}{4}}(M)$, $f_\epsilon \in L^\infty(M)$ such that

$$\begin{aligned} (\Delta_g + 1)^2 u &= \operatorname{div}_g (A^\# du) + (1 - a)u + 2\Delta_g u + \lambda f |u|^{2^\#-2} u \\ &= b + q_\epsilon u + f_\epsilon \end{aligned}$$

where $b = \operatorname{div}_g (A^\# du) + (1 - a)u + 2\Delta_g u$

According to theorem 2, for any $q > 1$ and any $f \in L^q(M)$, there exists a unique $u \in H_4^q(M)$ such that

$$(\Delta_g + 1)^2 u = f \text{ with } \|u\|_{H_4^q(M)} \leq \|f\|_{L^q(M)}.$$

Now, we consider the following operator

$$H_\epsilon : u \in L^q(M) \rightarrow (\Delta_g + 1)^{-2} (q_\epsilon u) \in L^q(M)$$

with

$$\begin{aligned} \|H_\epsilon u\|_q &= O\left(\left\|(\Delta_g + 1)^{-2} (q_\epsilon u)\right\|_q\right) = O\left(\left\|(\Delta_g + 1)^{-2} (q_\epsilon u)\right\|_{H_4^{\hat{q}}(M)}\right) \\ &\leq C \|q_\epsilon u\|_{\hat{q}} \leq C \|q_\epsilon\|_{\frac{n}{4}} \|u\|_q \leq C\epsilon \|u\|_q \end{aligned}$$

and where $\hat{q} = \frac{nq}{n+4s}$.

Hence, for $\epsilon > 0$ sufficiently small

$$\|H_\epsilon\|_{L^q \rightarrow L^q} \leq C\epsilon < \frac{1}{2}.$$

So, the operator

$$\begin{aligned} (Id - H_\epsilon) : L^q(M) &\rightarrow L^q(M) \\ (Id - H_\epsilon) u &= (\Delta_g + 1)^{-2} (b + f_\epsilon) \end{aligned}$$

is an invertible, and we get $b + f_\epsilon \in L^2(M)$, hence $(\Delta_g + 1)^{-2} (b + f_\epsilon) \in H_4^2(M)$.

By the Sobolev theorem, we deduce that

$$u \in L^{\frac{2n}{n-8}}(M), f |u|^{2^\#-2} u \in L^{\frac{2n}{(n-8)(2^\#-1)}}(M) = L^{\frac{2n(n-4)}{(n-8)(n+4)}}(M).$$

Since $\frac{2n(n-4)}{(n-8)(n+4)} > 2$, we obtain that

$$(\Delta_g + 1)^{-2} u \in L^2(M).$$

We now use a bootstrap argument, we construct an increasing sequence (p_i) such that $u \in H_4^{p_i}(M)$ for all $i \in \mathbb{N}$

We let $p_0 = 2$, the Sobolev's theorem asserts that

$$u \in L^{\frac{np_i}{n-4p_i}}(M) \text{ and } f|u|^{2^\sharp-2}u \in L^{\frac{np_i}{(n-4p_i)(2^\sharp-1)}}(M) = L^{\frac{np_i(n-4)}{(n-4p_i)(n+4)}}(M)$$

then

$$(\Delta_g + 1)^{-2} u \in L^{p_{i+1}}(M)$$

where

$$p_{i+1} = \begin{cases} \frac{np_i(n-4)}{(n-4p_i)(n+4)} & \text{if } p_i < \frac{n}{4} \\ +\infty & \text{if } p_i \geq \frac{n}{4} \end{cases}.$$

We can verify by recurrence that for all $i \in \mathbb{N}$ that $p_i > \frac{2n}{n+4}$, hence the sequence $(p_i)_i$ is increasing and bounded, consequently it converges to $l \geq 2$ fulfilling the relation

$$l = \frac{nl(n-4)}{(n-4l)(n+4)}.$$

Which gives $l = \frac{2n}{n+4}$; a contradiction, hence $p_i \rightarrow +\infty$ and $u \in H_4^p(M)$ for all $p > 1$. Applying again the Sobolev's theorem, we get $u \in C^{4,\alpha}(M)$ for some $\alpha \in (0, 1)$. \square

4. TEST FUNCTIONS

In this section, we prove that the insuring condition (3.6) in the proposition 2 is satisfied by using test functions.

For this we consider (y^1, y^2, \dots, y^n) a normal geodesic coordinate system centred at a point x_0 where f is maximal. Denote by $S(r)$ the geodesic sphere centred at x_0 and of radius r ($r < d =$ the injectivity radius). Let $d\Omega$ be the volume element of the Euclidean unit sphere $S^{n-1}(1)$ and put

$$G(r) = \frac{1}{\omega_{n-1}} \int_{S(r)} \sqrt{|g(x)|} d\Omega$$

where ω_{n-1} is the volume of $S^{n-1}(1)$ and $|g(x)|$ the determinant of the Riemannian metric g .

The Taylor's expansion of $G(r)$ in a neighborhood of $r = 0$ is given by

$$G(r) = 1 - \frac{R(x_0)}{6n} r^2 + o(r^2)$$

where $R(x_0)$ denotes the scalar curvature of M at x_0 .

Let $B(x_0, \delta)$ be the ball centred at x_0 and of radius δ such that $0 < 2\delta < d$ and let η be a smooth function equals to 1 on $B(x_0, \delta)$ and equals to 0 on $M - B(x_0, 2\delta)$.

We define the radial function

$$u_\epsilon(r) = \frac{\eta(r)}{(r^2 + \epsilon^2)^{\frac{n-4}{2}}}$$

where $r = d(x_0, x)$ is geodesic distance to the point x_0 .

To simplify the computations, we define the following functions, (see [2]) :

for any reel positive numbers p, q such that $p - q > 1$ we put

$$I_p^q = \int_0^{+\infty} \frac{t^q}{(1+t)^p} dt.$$

The following relations are immediate

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q, \quad I_{p+1}^{q+1} = \frac{q}{p-q-1} I_{p+1}^q.$$

First, we calculate the Taylor expansion of the quotient

$$Q_\epsilon = \frac{\mu(u_\epsilon)}{(\gamma(u_\epsilon))^{\frac{2}{2^\#}}}$$

where

$$\mu(u_\epsilon) = \int_M \left((\Delta_g u_\epsilon)^2 + A^\#(du_\epsilon, du_\epsilon) + au_\epsilon^2 \right) dv_g \text{ and } \gamma(u_\epsilon) = \int_M f |u_\epsilon|^{2^\#} dv_g.$$

We compute the different terms, as in [5, 9, 15] and in the case $n > 6$ we get

$$\int_M (\Delta_g u_\epsilon)^2 dv_g = \frac{(n-4)\omega_{n-1}I_n^{\frac{n}{2}-1}}{2\epsilon^{n-4}} \left[n(n^2-4) - \frac{n(n^2+4n-20)}{6(n-6)} R(x_0) \epsilon^2 + o(\epsilon^2) \right]$$

and also

$$\int_M A^\#(du_\epsilon, du_\epsilon) dv_g = \frac{(n-4)\omega_{n-1}I_n^{\frac{n}{2}-1}}{2\epsilon^{n-4}} \left(\frac{4(n-1)}{n-6} Tr_g A(x_0) \epsilon^2 + o(\epsilon^2) \right).$$

Easily we get

$$\int_M au_\epsilon^2 dv_g = \frac{1}{\epsilon^{n-4}} O(\epsilon^4).$$

By grouping the different terms of $\mu(u_\epsilon)$, we obtain

$$\mu(u_\epsilon) = \frac{n(n-4)(n^2-4)\omega_n}{2^n \epsilon^{n-4}} \times \left[1 + \frac{1}{n(n^2-4)(n-6)} \left(4Tr_g A(x_0)(n-1) - \frac{n(n^2+4n-20)}{6} R(x_0) \right) \epsilon^2 + o(\epsilon^2) \right]$$

where $\omega_n = 2^{n-1} I_n^{\frac{n}{2}-1} \omega_{n-1}$.

As in [5, 9, 15] and when $n > 6$, we have

$$\gamma(u_\epsilon) = \int_M f |u_\epsilon|^{2^\#} dv_g = \frac{f(x_0)\omega_n}{2^n \epsilon^n} \left\{ 1 - \frac{1}{6(n-2)} \left(\frac{3\Delta f(x_0)}{f(x_0)} + R(x_0) \right) \epsilon^2 + o(\epsilon^2) \right\}.$$

Hence

$$(\gamma(u_\epsilon))^{-\frac{2}{2^\#}} = \frac{(f(x_0))^{-\frac{2}{2^\#}} \omega_n^{-\frac{2}{2^\#}}}{2^{4-n} \epsilon^{4-n}} \left\{ 1 + \frac{n-4}{6n(n-2)} \left(R(x_0) + \frac{3\Delta f(x_0)}{f(x_0)} \right) \epsilon^2 + o(\epsilon^2) \right\}.$$

Finally the Taylor's expansion of Q_ϵ , when $n > 6$, is given by

$$Q_\epsilon = \frac{1}{(f(x_0))^{-\frac{2}{2^\sharp}} K_0} \left\{ \left(1 + \frac{1}{2n(n^2-4)(n-6)} \times \left((n+2)(n-4)(n-6) \frac{\Delta f(x_0)}{f(x_0)} + 8(n-1) \text{Tr}_g A(x_0) - 4(n^2-2n-4) R(x_0) \right) \right) \epsilon^2 + o(\epsilon^2) \right\}$$

where $\frac{1}{K_0} = \frac{n(n-4)(n^2-4)\omega_n^{\frac{4}{n}}}{16}$.

It is obvious that if

$$(n+2)(n-4)(n-6) \frac{\Delta f(x_0)}{f(x_0)} + 8(n-1) \text{Tr}_g A(x_0) - 4(n^2-2n-4) R(x_0) < 0$$

we have $Q_\epsilon < 1$

Now in the case $n = 6$, we have

$$\int_M (\Delta_g u_\epsilon)^2 dv_g = \frac{(n-4)^2 \omega_{n-1}}{2\epsilon^{n-4}} \left\{ \frac{n(n^2-4)}{n-4} I_n^{\frac{n}{2}-1} - \frac{2}{n} R(x_0) \epsilon^2 \ln \frac{1}{\epsilon^2} + o(\epsilon^2) \right\}$$

and also

$$\int_M A^\#(du_\epsilon, du_\epsilon) dv_g = \frac{(n-4)^2 \omega_{n-1} I_n^{\frac{n}{2}-1}}{2\epsilon^{n-4}} \left(\frac{\text{Tr}_g A(x_0)}{n} \epsilon^2 \ln \frac{1}{\epsilon^2} + o(\epsilon^2) \right)$$

The same calculations as in case $n > 6$, gives us

$$\int_M a u_\epsilon^2 dv_g = \frac{1}{\epsilon^{n-4}} O(\epsilon^4)$$

Grouping the terms of $\mu(u_\epsilon)$ we get

$$\begin{aligned} \mu(u_\epsilon) &= \int_M \left((\Delta_g u_\epsilon)^2 + A^\#(du_\epsilon, du_\epsilon) + a u_\epsilon^2 \right) dv_g \\ &= \frac{n(n-4)(n^2-4)\omega_n}{2^n \epsilon^{n-4}} \left\{ 1 + \frac{n-4}{(n^2-4) I_n^{\frac{n}{2}-1}} (\text{Tr}_g A(x_0) - 2R(x_0)) \epsilon^2 \ln \frac{1}{\epsilon^2} + o(\epsilon^2) \right\} \end{aligned}$$

where $\omega_n = 2^{n-1} I_n^{\frac{n}{2}-1} \omega_{n-1}$.

By the same calculations as in case $n > 6$ we obtain

$$(\gamma(u_\epsilon))^{-\frac{2}{2^\sharp}} = \frac{(f(x_0))^{-\frac{2}{2^\sharp}} \omega_n^{-\frac{2}{2^\sharp}}}{2^{4-n} \epsilon^{4-n}} \left\{ 1 + \frac{n-4}{6n(n-2)} \left(R(x_0) + \frac{3\Delta f(x_0)}{f(x_0)} \right) \epsilon^2 + o(\epsilon^2) \right\}.$$

Finally the Taylor expansion of Q_ϵ , when $n = 6$, is given by

$$Q_\epsilon = \frac{1}{(f(x_0))^{-\frac{2}{2^\sharp}} K_0} \left\{ 1 + \frac{n-4}{(n^2-4) I_n^{\frac{n}{2}-1}} (\text{Tr}_g A(x_0) - 2R(x_0)) \epsilon^2 \ln \frac{1}{\epsilon^2} + o(\epsilon^2) \right\}.$$

And assuming $\text{Tr}_g A(x_0) < 2R(x_0)$, we get that $Q_\epsilon < 1$.

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